

PRESENTATIONS OF CHARACTER VARIETIES OF 2-BRIDGE KNOTS USING CHEBYSHEV POLYNOMIALS

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ABSTRACT. In this paper, we use Chebyshev polynomials to give presentations of the character varieties of certain types of 2-bridge knots. This gives us an elementary method using basic calculations to discuss the number of irreducible components of the character varieties and thus to recover the results of Burde on the irreducibility of non-abelian $SU(2)$ -representation spaces in [2]. These results can be applied to determine some minimal elements of a partial ordering of prime knots.

1. INTRODUCTION

In 1980' Culler and Shalen introduced in [5] an algebraic set in a complex space for a finitely presented group G , now known as the character variety of G . The framework of character varieties has been giving powerful tools and is now playing important roles in geometry and topology. On the other hand, it is not easy to calculate character varieties and thus to investigate the geometric structures in general, though an underlying idea of character varieties is simple as follows. Let G be a finitely presented group generated by n elements g_1, \dots, g_n . For a representation $\rho : G \rightarrow \mathrm{SL}_2(\mathbb{C})$, the character χ_ρ of ρ is the function on G defined by $\chi_\rho(g) := \mathrm{tr}(\rho(g))$ ($\forall g \in G$). Throughout this paper, we simply denote by $\mathrm{tr}(g)$ the trace $\mathrm{tr}(\rho(g))$ for an unspecified representation $\rho : G \rightarrow \mathrm{SL}_2(\mathbb{C})$. We sometimes omit the brackets in the trace like $\mathrm{tr}(a) = \mathrm{tr} a$ for simplicity. By [5] (see also [7]), the $\mathrm{SL}_2(\mathbb{C})$ -trace identity

$$\mathrm{tr}(AB) = \mathrm{tr}(A)\mathrm{tr}(B) - \mathrm{tr}(AB^{-1}) \quad (A, B \in \mathrm{SL}_2(\mathbb{C}))$$

shows that $\mathrm{tr}(g)$ is expressed by a polynomial in $\{\mathrm{tr}(g_i)\}_{1 \leq i \leq n}$, $\{\mathrm{tr}(g_i g_j)\}_{1 \leq i < j \leq n}$ and $\{\mathrm{tr}(g_i g_j g_k)\}_{1 \leq i < j < k \leq n}$, for any $g \in G$. Then the character variety of G , denoted by $X(G)$, is constructed basically by the image of the set of characters $\chi(G)$ of $\mathrm{SL}_2(\mathbb{C})$ -representations of G under the map¹

$$t : \chi(G) \rightarrow \mathbb{C}^{n + \binom{n}{2} + \binom{n}{3}}, \quad t(\chi_\rho) := (\mathrm{tr}(g_i); \mathrm{tr}(g_i g_j); \mathrm{tr}(g_i g_j g_k)).$$

The resulting set turns out to be a closed algebraic set (refer to [5]). By definition, this algebraic set depends on a choice of generators of G (the coordinates of $X(G)$ vary if we change the choice of generating set of G), however, the geometric structures do not depend on that choice. So $X(G)$ is an invariant of G up to *isomorphism of algebraic sets*. Here two algebraic sets V and W in a complex space \mathbb{C}^N are said to be isomorphic or *bipolynomial* if there exist polynomial maps $f : V \rightarrow W$ and $g : W \rightarrow V$ such that $g \circ f = \mathrm{id}_V$, $f \circ g = \mathrm{id}_W$. Then f and g are called isomorphisms or bipolynomial maps.

As far as the authors know, there exist few discussions on isomorphic transformations of character varieties. In this paper, we first give two polynomial maps giving isomorphisms

¹The map t is not injective, however, for the characters of irreducible representations it is injective. Refer to [5].

between character varieties of a group G coming from two different choices of generating sets, especially in the case where G is the fundamental group of a 3-manifold associated to a *knot* in 3-sphere as follows. Let \mathbb{S}^3 denote a 3-sphere. For a knot K in \mathbb{S}^3 , which is an embedding of a circle \mathbb{S}^1 in \mathbb{S}^3 , we denote by $G(K)$ the knot group of K , i.e., the fundamental group of the knot complement $\mathbb{S}^3 - K$. For example, there exist knots parametrized by sequences of integers $[a_1, a_2, \dots, a_r]$ associated to the number of *twists*, called *2-bridge knots*. (See Figure 1. For more details, refer to [9].) The knot $K_m = [2, m]$ depicted in Figure 2 is a type of 2-bridge knots, called the *m-twist knot*. Sometimes the sequence $[a_1, a_2, \dots, a_r]$ is encoded in a rational number p/q ($p > 0, -p < q < p$) by the following continuous fraction:

$$\frac{p}{q} = a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_r}}}.$$

Then we denote by $\mathbf{b}(p, q)$ the 2-bridge knot with a rational number expression p/q . It is shown that p and q are coprime odd integers (refer to [9]).

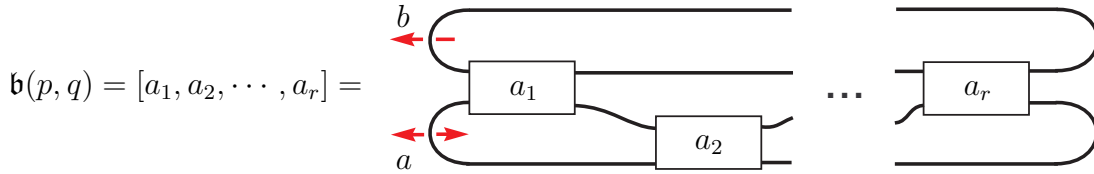


FIGURE 1. The 2-bridge knot $\mathbf{b}(p, q) = [a_1, a_2, \dots, a_r]$ and generators a and b of $G(\mathbf{b}(p, q))$ for r odd. The orientation of a is chosen so that a and b are conjugate. a_i denotes the number of twists with *sign* in the white box.

By *Wirtinger's algorithm* and reduction of generators, we have the following presentation of the knot group $G(\mathbf{b}(p, q))$:

$$G(\mathbf{b}(p, q)) = \langle a, b \mid wa = bw \rangle,$$

where a, b are *meridians* shown in Figure 1, $w = a^{\varepsilon_1} b^{\varepsilon_2} \dots a^{\varepsilon_{p-2}} b^{\varepsilon_{p-1}}$ and $\varepsilon_j = (-1)^{\lfloor jq/p \rfloor}$. Here $\lfloor s \rfloor$ denotes the maximal integer m satisfying $m \leq s$. By [11], the character variety $X(\mathbf{b}(p, q)) := X(G(\mathbf{b}(p, q)))$ is given as the algebraic set defined by

$$\mathrm{tr}(bwa^{-1}) - \mathrm{tr}(w) = 0.$$

For example, this equation induces the following presentation of the character variety $X(K_{2n}) := X(G(K_{2n}))$ of the $2n$ -twist knot K_{2n} . Let $S_n(z)$ be the Chebyshev polynomial of the second kind, i.e., for any integer n ,

$$S_n(z) = zS_{n-1}(z) - S_{n-2}(z), \quad S_1(z) = z, \quad S_0(z) = 1.$$

Note that $S_{-n}(z) = -S_{n-2}(z)$ holds for any integer n . We denote by $L_n(x, y)$ the polynomial in the complex polynomial ring $\mathbb{C}[x, y]$ defined by

$$L_n(x, y) = (y - 2) (S_n(t) + (y + 1 - x^2)S_{n-1}(t)),$$

where $t = y^2 - yx^2 + 2x^2 - 2$. Using $L_n(x, y)$, we can describe the character variety $X(K_{2n})$ as the algebraic set

$$(1.1) \quad X(K_{2n}) = \{(\bar{x}, \bar{y}) \in \mathbb{C} \mid L_n(\bar{x}, \bar{y}) = 0\},$$

where $\bar{x} = \text{tr}(a) = \text{tr}(b)$ and $\bar{y} = \text{tr}(ab^{-1}) = \text{tr}(\bar{y}_*)$. (See Subsection 2.1 for details. For K_{2n-1} , refer to Subsection 2.2.)

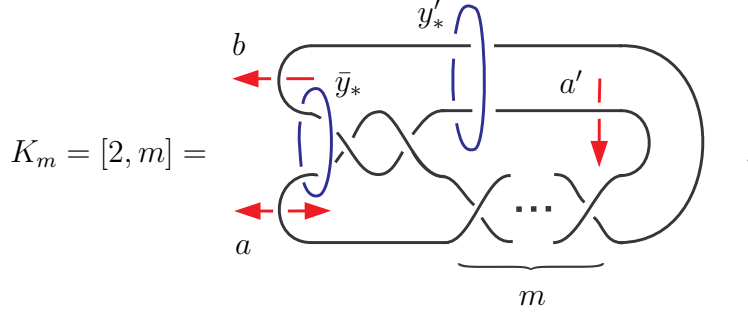


FIGURE 2. The m -twist knot $K_m = [2, m] = \mathfrak{b}(2m + 1, m)$ and loops \bar{y}_* and y'_* parametrizing $X(K_m)$. The orientation of a is chosen so that a and b are conjugate.

On the other hand, we can also calculate the character variety using *the Kauffman bracket skein module*. (Refer to [1, 18, 19, 20]. See also Subsection 2.3.) From the Kauffman bracket skein theory, the following is a basic presentation of $X(K_m) := X(G(K_m))$. For any integer m , let $R_m(x, y)$ be the polynomial in $\mathbb{C}[x, y]$ defined by

$$R_m(x, y) = (y + 2) \left(S_m(y) - S_{m-1}(y) + x^2 \sum_{i=0}^{m-1} S_i(y) \right),$$

and let $\tilde{R}_m(x, y)$ be the second factor of $R_m(x, y)$. Then the character variety $X(K_m)$ is also described as the algebraic set

$$(1.2) \quad X(K_m) = \{(x', y') \in \mathbb{C}^2 \mid R_m(x', y') = 0\},$$

where $x' = -\text{tr}(a') = -\text{tr}(b)$ and $y' = -\text{tr}(a'b^{-1}) = -\text{tr}(y'_*)$ (refer to [6]). By definition, the algebraic sets in (1.1) and (1.2) at $m = 2n$ must be isomorphic as algebraic sets. In Subsection 2.1, we construct bipolynomial maps which give isomorphisms between (1.1) and (1.2) at $m = 2n$.

Actually, the presentation in (1.2) of $X(K_m)$ is more useful than (1.1) to discuss a geometric property of $X(G)$, that is, the number of irreducible components. For example, the presentation (1.2) of $X(K_m)$ immediately shows the following result.

Theorem 1.1 (cf. [2, 13]). *For any positive integer² m , the polynomial $\tilde{R}_m(x, y)$ is irreducible in $\mathbb{C}[x, y]$. Therefore, $X(K_m)$ consists of two irreducible components.*

²For a negative integer $-m$ ($m > 1$), taking the mirror image of K_{-m} and arranging it, we can obtain $X(K_{-m}) = X(K_{m-1})$ and thus a similar result to Theorem 1.1. In that case, $R_{-m}(x, y)$ will shift to $R_{m-1}(x, y)$.

Indeed, as in [15] the factor $\tilde{R}_m(x, y)$

$$\begin{aligned}\tilde{R}_m(x, y) &= S_m(y) - S_{m-1}(y) + x^2 \sum_{i=0}^{m-1} S_i(y) \\ &= S_m(y) - S_{m-1}(y) + x^2 \frac{S_m(y) - S_{m-1}(y) - 1}{y - 2}\end{aligned}$$

cannot be factorized as $(h_1x + h_2)(h_3x + h_4)$ where $h_j \in \mathbb{C}[y]$. Moreover, $(h_1x^2 + h_2)h_3$ where $h_j \in \mathbb{C}[y]$ cannot occur either as a factorization of $\tilde{R}_m(z, y)$, since $S_m(y) - S_{m-1}(y)$ and $(S_m(y) - S_{m-1}(y) - 1)/(y - 2)$ are relatively prime in $\mathbb{C}[y]$. Hence $\tilde{R}_m(x, y)$ is irreducible in $\mathbb{C}[x, y]$ and this completes the proof of Theorem 1.1.

It was studied in [13] that the character variety of most *double twist knots*, including twist knots, consists of two irreducible components, however, they used the genera of algebraic varieties to show the fact. Our proof of Theorem 1.1 shown above uses only basic calculations on the Chebyshev polynomials $S_n(z)$. This is an efficiency of Chebyshev polynomials.

In fact, the method used in the proof of Theorem 1.1 can be applied to another type of 2-bridge knots denoted by $\mathfrak{b}(p, 3)$, which was considered in [2]. Then that gives the following theorem using basic calculations.

Theorem 1.2 (Theorem 3.3 in Section 3, c.f. [2]). *The character variety $X(\mathfrak{b}(p, 3))$ consists of exactly two irreducible components.*

This paper mainly shows

- presentations of the character varieties of 2-bridge knots K_m and $\mathfrak{b}(p, 3)$ using the Chebyshev polynomials $S_n(z)$, including the transformations between the algebraic sets in (1.1) and (1.2),
- an elementary proof of Theorem 1.2 using Chebyshev polynomials.

These recover the results of Burde on the irreducibility of non-abelian $SU(2)$ -representation spaces in [2]. Moreover, they can be applied to determine some minimal elements of a partial ordering of prime knots (see Section 4).

2. CHARACTER VARIETIES OF TWIST KNOTS

2.1. Expressions of $X(K_{2n})$ for twist knots K_{2n} . For the twist knot $K_{2n} = \mathfrak{b}(4n + 1, 2n + 1)$, we have $w = u^n$ where $u = ab^{-1}a^{-1}b$. Let $x := \text{tr}(a) = \text{tr}(b)$, $y := \text{tr}(ab^{-1}) = \text{tr}(\bar{y}_*)$ and $t := t(x, y) = \text{tr} u = \text{tr}(ab^{-1}a^{-1}b) = \text{tr}(aba^{-1}b^{-1}) = y^2 - yx^2 + 2x^2 - 2$. We first focus on the Chebyshev polynomials $\{S_n(t)\}_{n=-\infty}^{\infty}$.

The following lemma is standard, see e.g. [23, Lemma 2.2].

Lemma 2.1. *Suppose the sequence $\{f_n\}_{n=-\infty}^{\infty}$ satisfies the recurrence relation $f_{n+1} = tf_n - f_{n-1}$. Then $f_n = f_0 S_n(t) - f_{-1} S_{n-1}(t)$.*

Applying Lemma 2.1, we have

$$\begin{aligned}\text{tr} bwa^{-1} - \text{tr} w &= \text{tr} bu^n a^{-1} - \text{tr} u^n \\ &= (\text{tr} ba^{-1} - \text{tr} 1)S_n(t) - (\text{tr} bu^{-1}a^{-1} - \text{tr} u^{-1})S_{n-1}(t) \\ &= (y - 2)S_n(t) - (y - t)S_{n-1}(t) \\ &= (y - 2)(S_n(t) + (y + 1 - x^2)S_{n-1}(t)),\end{aligned}$$

since $t - y = (y - 2)(y + 1 - x^2)$. Therefore the character variety $X(K_{2n})$ is given by the presentation in (1.1).

Now we construct two polynomial maps f and g that give isomorphisms between the algebraic sets in (1.1) and (1.2). Define a polynomial $X_m := X_m(x, y)$ in $\mathbb{C}[x, y]$ by

$$(2.1) \quad X_0 = -2, \quad X_1 = -x^2 - y, \quad X_{m+1} - yX_m + X_{m-1} + 2x^2 = 0.$$

In fact, by [6] the general term of X_m is

$$(2.2) \quad X_m = -S_m(y) + S_{m-2}(y) - x^2 \left(S_{m-1}(y) + 2 \sum_{i=0}^{m-2} S_i(y) \right)$$

and then we can check that $R_m(x, y) = -(X_{m+1} + X_m + x^2)$ holds.

Now let $f, g : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ be the maps

$$\begin{aligned} f(x, y) &= (-x, -t(x, y)), \\ g(x, y) &= (-x, -X_{2n}(x, y)). \end{aligned}$$

Then we obtain

$$\begin{aligned} (f \circ g)(x, y) &= (x, -X_{2n}(-x, -t(x, y))), \\ (g \circ f)(x, y) &= (x, -t(-x, -X_{2n}(x, y))). \end{aligned}$$

We will show that $(f \circ g)(x, y) = (x, y)$ on the algebraic set (1.2) at $m = 2n$ and $(g \circ f)(x, y) = (x, y)$ on (1.1). To prove these, we will need the following lemmas.

Lemma 2.2 (Lemma 4.3 in [14]). *For any non-negative integers r and s ,*

$$S_r(z)S_{r+s}(z) = S_{2r+s}(z) + S_{2r+s-2}(z) + \cdots + S_s(z).$$

Lemma 2.2 and $S_{-m}(z) = -S_{m-2}(z)$ immediately show Lemmas 2.3 and 2.4.

Lemma 2.3. $S_m(z)S_{m-2}(z) - S_{m-1}(z)^2 = -1$ holds for any integer m .

Lemma 2.4. $S_{m+1}(z)^2 + S_{m-1}(z)^2 - (z^2 - 2)S_m(z)^2 = 2$ holds for any integer m .

Lemma 2.5. *One has*

$$X_m^2 + X_{m-1}^2 - yX_mX_{m-1} + 2x^2(X_m + X_{m-1}) = -y^2 - 2x^2y - x^4 - 4x^2 + 4.$$

Proof. Let $\beta_m = X_m^2 + X_{m-1}^2 - yX_mX_{m-1} + 2x^2(X_m + X_{m-1})$. Then we have

$$\begin{aligned} \beta_{m+1} - \beta_m &= (X_{m+1}^2 - X_{m-1}^2) - yX_m(X_{m+1} - X_{m-1}) + 2x^2(X_{m+1} - X_{m-1}) \\ &= (X_{m+1} - X_{m-1})(X_{m+1} + X_{m-1} - yX_m + 2x^2) = 0. \end{aligned}$$

Hence $\beta_m = \beta_1 = -y^2 - 2x^2y - x^4 - 4x^2 + 4$. □

Lemma 2.6. *One has*

$$R_m(x, y)R_{m-1}(x, y) = (y + 2)(X_m^2 + x^2X_m + y + 2x^2 - 2).$$

Proof. We have

$$\begin{aligned}
& R_m(x, y)R_{m-1}(x, y) \\
&= (X_{m+1} + X_m + x^2)(X_m + X_{m-1} + x^2) \\
&= ((y+1)X_m - X_{m-1} - x^2)(X_m + X_{m-1} + x^2) \\
&= (y+1)X_m^2 + yX_m(X_{m-1} + x^2) - (X_{m-1} + x^2)^2 \\
&= (y+2)(X_m^2 + x^2X_m) - (X_m^2 + X_{m-1}^2 - yX_mX_{m-1} + 2x^2(X_m + X_{m-1})) - x^4 \\
&= (y+2)(X_m^2 + x^2X_m) + (y^2 + 2x^2y + 4x^2 - 4)
\end{aligned}$$

by Lemma 2.5. □

Lemma 2.7. *One has $X_{m+2} + X_{m-2} - (y^2 - 2)X_m = -(2y + 4)x^2$.*

Proof. We have

$$\begin{aligned}
X_{m+2} + X_{m-2} &= (yX_{m+1} - X_m - 2x^2) + (yX_{m-1} - X_m - 2x^2) \\
&= y(X_{m+1} + X_{m-1}) - 2X_m - 4x^2 \\
&= y(yX_m - 2x^2) - 2X_m - 4x^2 \\
&= (y^2 - 2)X_m - (2y + 4)x^2.
\end{aligned}$$

Hence $X_{m+2} + X_{m-2} - (y^2 - 2)X_m = -(2y + 4)x^2$. □

Now we can show one of the desired propositions. Note that $-t(-x, -X_m(x, y)) = -X_m(x, y)^2 - x^2X_m(x, y) - 2x^2 + 2$.

Proposition 2.8. *The following holds for any integer m .*

$$-X_m(x, y)^2 - x^2X_m(x, y) - 2x^2 + 2 = y - (y+2)\tilde{R}_m(x, y)\tilde{R}_{m-1}(x, y).$$

Hence $(g \circ f)(x, y) = (x, y) \pmod{R_{2n}(x, y)}$.

Proof. This is equivalent to Lemma 2.6 which says that

$$R_m(x, y)R_{m-1}(x, y) = (y+2)(X_m^2 + x^2X_m + y + 2x^2 - 2).$$

The proposition follows. □

We can also show the remaining desired proposition.

Proposition 2.9. *The following holds for any integer n .*

$$-X_{2n}(-x, -t) - y = (y-2) \left(S_n(t) + (y+1-x^2)S_{n-1}(t) \right) \left((y+1-x^2)S_{n-1}(t) + S_{n-2}(t) \right).$$

Hence $(f \circ g)(x, y) = (x, y) \pmod{L_n(x, y)}$.

Proof. Let

$$\gamma_n(x, y) = \left(S_n(t) + (y+1-x^2)S_{n-1}(t) \right) \left((y+1-x^2)S_{n-1}(t) + S_{n-2}(t) \right).$$

By direct calculations one can check that $-X_{2n}(-x, -t) - y = (y-2)\gamma_n$ for $n = 0, 1$. By Lemma 2.7, we have

$$X_{m+2}(-x, -t) + X_{m-2}(-x, -t) - (t^2 - 2)X_m = (2t - 4)x^2.$$

It follows that

$$\begin{aligned}
& (X_{m+2}(-x, -t) + y) + (X_{m-2}(-x, -t) + y) - (t^2 - 2)(X_m + y) \\
&= (2t - 4)x^2 - (t^2 - 4)y \\
&= (t - 2)(2x^2 - (t + 2)y) \\
&= (y - 2)(y + 2 - x^2)(2x^2 - (t + 2)y),
\end{aligned}$$

since $t - 2 = (y - 2)(y + 2 - x^2)$. To prove Proposition 2.9, it suffices to show that

$$(2.3) \quad \gamma_{n+1} + \gamma_{n-1} - (t^2 - 2)\gamma_n = -(y + 2 - x^2)(2x^2 - (t + 2)y).$$

By Lemma 2.3, we obtain

$$\begin{aligned}
\gamma_n &= (S_n(t) + (y + 1 - x^2)S_{n-1}(t)) ((y + 1 - x^2)S_{n-1}(t) + S_{n-2}(t)) \\
&= (y + 1 - x^2)S_{n-1}(t) (S_n(t) + S_{n-2}(t) + (y + 1 - x^2)S_{n-1}(t)) + S_n(t)S_{n-2}(t) \\
&= (y + 1 - x^2)(t + y + 1 - x^2)S_{n-1}^2(t) + S_{n-1}^2(t) - 1 \\
&= [(y + 1 - x^2)(t + y + 1 - x^2) + 1]S_{n-1}^2(t) - 1,
\end{aligned}$$

Let $\delta = (y + 1 - x^2)(t + y + 1 - x^2) + 1$. Then $\gamma_n = \delta S_{n-1}^2(t) - 1$. Since by Lemma 2.4 $S_{n+1}^2(t) + S_{n-1}^2(t) - (t^2 - 2)S_n^2(t) = 2$, it follows that

$$\begin{aligned}
\gamma_{n+1} + \gamma_{n-1} - (t^2 - 2)\gamma_n &= \delta (S_{n+1}^2(t) + S_{n-1}^2(t) - (t^2 - 2)S_n^2(t)) + t^2 - 4 \\
&= 2\delta + t^2 - 4
\end{aligned}$$

It is easy to check that $2\delta + t^2 - 4 = -(y + 2 - x^2)(2x^2 - (t + 2)y)$. Hence Eq. (2.3) holds true for all n . This completes the proof Proposition 2.9. \square

2.2. Expressions of $X(K_{2n-1})$ for twist knots K_{2n-1} . For the twist knot $K_{2n-1} = \mathbf{b}(4n - 1, 2n - 1)$, $w = v^{n-1}ab$ where $v = aba^{-1}b^{-1}$. In this case, we let $x = \text{tr}(a) = \text{tr}(b)$, $y = \text{tr}(ab) = \text{tr}(\bar{y}_*)$ and $t := t(x, y) = \text{tr}(v) = \text{tr}(a^{-1}bab^{-1}) = \text{tr}(aba^{-1}b^{-1}) = y^2 - yx^2 + 2x^2 - 2$. Then as in the case of K_{2n} it follows from Lemma 2.1 that

$$\begin{aligned}
\text{tr } bwa^{-1} - \text{tr } w &= \text{tr } bv^{n-1}aba^{-1} - \text{tr } v^{n-1}ab \\
&= (\text{tr } baba^{-1} - \text{tr } ab)S_{n-1}(t) - (\text{tr } bv^{-1}aba^{-1} - \text{tr } v^{-1}ab)S_{n-2}(t) \\
&= ((x^2 - t) - y)S_{n-1}(t) - (\text{tr } b^2 - \text{tr } ba)S_{n-2}(t) \\
&= (x^2 - t - y)S_{n-1}(t) - ((x^2 - 2) - y)S_{n-2}(t) \\
&= (x^2 - y - 2)((y - 1)S_{n-1}(t) - S_{n-2}(t)).
\end{aligned}$$

So the character variety $X(K_{2n-1})$ is described as the algebraic set

$$(2.4) \quad X(K_{2n-1}) = \{(x, y) \in \mathbb{C}^2 \mid L'_n(x, y) = 0\},$$

where $L'_n(x, y) := (x^2 - y - 2)((y - 1)S_{n-1}(t) - S_{n-2}(t))$.

Now let $f, g : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ be the maps

$$\begin{aligned}
f(x, y) &= (-x, -t(x, y)), \\
g(x, y) &= (-x, -X_{2n-1}(x, y)).
\end{aligned}$$

Then we have

$$\begin{aligned}
(f \circ g)(x, y) &= (x, -X_{2n-1}(-x, -t(x, y))), \\
(g \circ f)(x, y) &= (x, -t(-x, -X_{2n-1}(x, y))).
\end{aligned}$$

Note that $-t(-x, -X_m(x, y)) = -X_m(x, y)^2 - x^2 X_m(x, y) - 2x^2 + 2$ and thus by Proposition 2.8 the equation $(g \circ f)(x, y) = (x, y)$ holds on the algebraic set in (1.2). So we will show the remaining equation $(f \circ g)(x, y) = (x, y)$ on the algebraic set in (2.4) at $m = 2n - 1$.

Proposition 2.10. *The following holds for any integer n .*

$$-X_{2n+1}(-x, -t) - y = -(x^2 - y - 2)(S_n(t) + (1 - y)S_{n-1}(t))((1 - y)S_n(t) + S_{n-1}(t)).$$

Hence $(f \circ g)(x, y) = (x, y) \pmod{L'_n(x, y)}$.

Proof. The proof is similar to that of Proposition 2.9. Let

$$\gamma'_n(x, y) = (S_n(t) + (1 - y)S_{n-1}(t))((1 - y)S_n(t) + S_{n-1}(t)).$$

By direct calculation, one can check that $-X_{2n+1}(-x, -t) - y = -(x^2 - y - 2)\gamma'_n$ for $n = 0, 1$. Since by Lemma 2.7

$$X_{m+2}(-x, -t) + X_{m-2}(-x, -t) - (t^2 - 2)X_m = (2t - 4)x^2,$$

the following holds

$$\begin{aligned} & (X_{m+2}(-x, -t) + y) + (X_{m-2}(-x, -t) + y) - (t^2 - 2)(X_m(-x, -t) + y) \\ &= (2t - 4)x^2 + (t^2 - 4)y \\ &= (y - 2)(y + 2 - x^2)(2x^2 - (t + 2)y) \end{aligned}$$

To prove Proposition 2.10, it suffices to show that

$$(2.5) \quad \gamma'_{n+1} + \gamma'_{n-1} - (t^2 - 2)\gamma'_n = (y + 2 - x^2)(2x^2 - (t + 2)y).$$

By Lemma 2.3, we have

$$\begin{aligned} \gamma'_n &= (S_n(t) + (1 - y)S_{n-1}(t))((1 - y)S_n(t) + S_{n-1}(t)) \\ &= (1 - y)(S_n^2(t) + S_{n-1}^2(t)) + (1 + (1 - y)^2)S_n(t)S_{n-1}(t) \\ &= (1 - y)(1 + tS_n(t)S_{n-1}(t)) + (1 + (1 - y)^2)S_n(t)S_{n-1}(t) \\ &= [(1 - y)(t + 1 - y) + 1]S_n(t)S_{n-1}(t) + 1 - y, \end{aligned}$$

Let $\delta = (1 - y)(t + 1 - y) + 1$. Then $\gamma'_n = \delta S_n(t)S_{n-1}(t) + 1 - y$. Since by Lemma 2.2 $S_{n+1}(t)S_n(t) + S_{n-1}(t)S_{n-2}(t) - (t^2 - 2)S_n(t)S_{n-1}(t) = t$, it follows that

$$\begin{aligned} & \gamma'_{n+1} + \gamma'_{n-1} - (t^2 - 2)\gamma'_n \\ &= \delta(S_{n+1}(t)S_n(t) + S_{n-1}(t)S_{n-2}(t) - (t^2 - 2)S_n(t)S_{n-1}(t)) - (t^2 - 4)(1 - y) \\ &= t\delta - (t^2 - 4)(1 - y). \end{aligned}$$

It is not hard to check that $t\delta - (t^2 - 4)(1 - y) = (y + 2 - x^2)(2x^2 - (t + 2)y)$. Hence Eq. (2.5) holds true for all n . Proposition 2.10 follows. \square

2.3. A topological background of the maps f and g . The difference between the presentations in (1.1) and (1.2) or (2.4) and (1.2) is the method to calculate the character varieties. As seen in Subsections 2.1 and 2.2, the presentations (1.1) and (2.4) are calculated along the definition using the trace functions of representations. On the other hand, the presentation in (1.2) is calculated by using the Kauffman bracket skein algebra³ (KBSA for short). The KBSA of a 3-manifold is the quotient of the module over

³This is the specialization of the Kauffman bracket skein module at the parameter $t = -1$.

\mathbb{C} generated by all free homotopy classes of loops in M by the *Kauffman bracket skein relations*:

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = - \begin{array}{c} \diagup \\ \diagdown \end{array} - \begin{array}{c} \diagdown \\ \diagup \end{array}, \quad \bigcirc = -2,$$

where in the first relation loops coincide each other outside dashed circles (refer to [1, 18, 19, 20]). By the result of [1] or [20], a loop s can be considered as $-\text{tr}(s)$ and the *Kauffman bracket skein relations* can be thought of as the $\text{SL}_2(\mathbb{C})$ -trace identities. This correspondence gives a method to calculate character varieties using the Kauffman bracket skein theory.

From the KBSA point of view, we can take two free homotopy classes of loops represented by b and y'_* as the generators of the KBSA of $\mathbb{S}^3 - K_m$ (see [6, Section 1]). These give the parameters $x' = -\text{tr}(a') = -\text{tr}(b)$ and $y' = -\text{tr}(y'_*) = -\text{tr}(aba^{-1}b^{-1}) = -\text{tr}(a'b^{-1})$ of the presentation in (1.2) and thus $X(K_m)$. If we want to transform the algebraic set in (1.1) or (2.4) to (1.2), we need to express the parameters $\bar{x} = \text{tr}(a) = \text{tr}(b)$ and $\bar{y} = \text{tr}(\bar{y}_*)$ in (1.1) or (2.4) as polynomials in x' and y' . The parameter \bar{x} is exactly $-x'$. On the other hand, \bar{y} is $\text{tr}(X_m^*)$ for the loop X_m^* shown in Figure 3 freely homotopic to \bar{y}_* . In the KBSA, $-\text{tr}(X_m^*)$ is considered as the loop X_m^* and then it follows from the Kauffman bracket skein relations that $X_0^* = -2$, $X_1^* = -(x'_*)^2 - y'_*$ and $X_{m+1}^* - y'_* X_m^* + X_{m-1}^* + 2(x'_*)^2 = 0$. (See [6]. Note that the loop X_m defined in [6] is actually the mirror image of the above X_m^* , however, the recursion relation does not change under the mirror image.) The general term X_m^* is the polynomial shown in (2.2) substituted $x = x'_*$ and $y = y'_*$. Therefore we defined $X_m := X_m(x, y)$ as in (2.1) and the map $f : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ by $f(x, y) = (-x, -X_m(x, y))$, which comes from $(\bar{x}, \bar{y}) \mapsto (-x', -X_m^*)$.

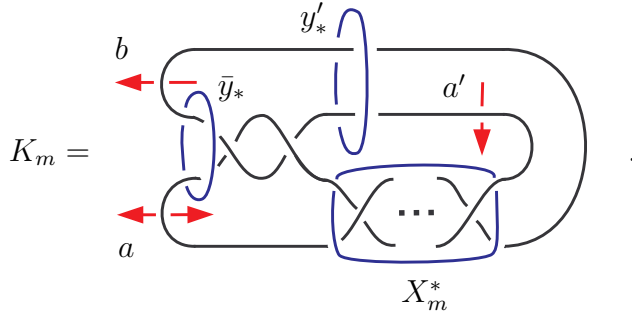


FIGURE 3. Loop X_m^* freely homotopic to \bar{y}_* .

Conversely, if we want to transform the algebraic set in (1.2) to (1.1) or (2.4), we need to express the parameters $x' = -\text{tr}(a') = -\text{tr}(b)$ and $y' = -\text{tr}(y'_*) = -\text{tr}(a'b^{-1})$ of (1.2) as polynomials in $\bar{x} = \text{tr}(a) = \text{tr}(b)$ and $\bar{y} = \text{tr}(\bar{y}_*)$. By the same reason, x' is exactly $-x$, and $-y'$ is $t := \text{tr}(aba^{-1}b^{-1}) = \bar{y}^2 - \bar{y}\bar{x}^2 + 2\bar{x}^2 - 2$. Therefore we define the map $g : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ by $g(x, y) = (-x, -t(x, y))$, where $t(x, y) = y^2 - yx^2 + 2x^2 - 2$. This map comes from $(x', y') \mapsto (-\bar{x}, -t)$.

3. CHARACTER VARIETIES OF 2-BRIDGE KNOTS $\mathfrak{b}(p, 3)$

Let $z = \text{tr}(ab)$ and $d = (p - 1)/2$. In general, it follows from [11] that the polynomial $\text{tr}(bwa^{-1}) - \text{tr}(w)$, whose zero set coincides with the character variety $X(\mathfrak{b}(p, m))$, is described by

$$\text{tr}(bwa^{-1}) - \text{tr}(w) = (z + 2 - x^2)\Phi_w(x, z),$$

where $\Phi_w(x, z)$ is the polynomial in $\mathbb{C}[x, z]$ defined by

$$\Phi_w(x, z) = \text{tr } w - \text{tr } w' + \cdots + (-1)^{d-1} \text{tr } w^{(d-1)} + (-1)^d.$$

Here if u is a word then u' denotes the word obtained from u by deleting the two letters at the two ends. Then $w^{(d-1)}$ means the element obtained from w by applying the deleting operation $d - 1$ times.

The above presentation of $X(\mathfrak{b}(p, m))$ is useful to show the irreducibility of $\Phi_w(x, z)$ over \mathbb{C} in the case of $\mathfrak{b}(p, 3)$.

Proposition 3.1. *For the 2-bridge knot $\mathfrak{b}(p, 3)$, one has*

$$\Phi_w(x, z) = S_d(z) - S_{d-1}(z) + x^2(2 - z)S_{d-\ell-1}(z)S_{\ell-1-\lfloor \frac{\ell}{2} \rfloor}(z) \left(S_{\lfloor \frac{\ell}{2} \rfloor}(z) - S_{\lfloor \frac{\ell}{2} \rfloor-1}(z) \right),$$

where $\ell = \lfloor \frac{p}{3} \rfloor$.

The proof of Proposition 3.1 will be presented at the end of this section. We will need the following result, which is a generalization of the method used in [15] and the proof of Theorem 1.1, in order to show the irreducibility of $\Phi_w(x, z)$ over \mathbb{C} in the case of $\mathfrak{b}(p, 3)$.

Lemma 3.2. *Suppose $\Phi(x, z) = f(z) + x^2g(z)$ is a polynomial in $\mathbb{C}[x, z]$ such that $\deg f - \deg g$ is an odd number, and $f(z)$ and $g(z)$ are relatively prime in $\mathbb{C}[z]$. Then $\Phi(x, z)$ is irreducible in $\mathbb{C}[x, z]$.*

Proof. Assume $\Phi(x, z)$ is reducible in $\mathbb{C}[x, z]$. Since $\gcd(f(z), g(z)) = 1$, we must have

$$(3.1) \quad \Phi(x, z) = (h_1(z) + xh_2(z))(h_3(z) + xh_4(z)),$$

where h_j 's are polynomials in $\mathbb{C}[z]$. Eq. (3.1) is equivalent to

$$f(z) = h_1(z)h_3(z), \quad 0 = h_1(z)h_4(z) + h_2(z)h_3(z), \quad g(z) = h_2(z)h_4(z).$$

It follows that

$$\deg f = \deg h_1 + \deg h_3, \quad \deg h_1 + \deg h_4 = \deg h_2 + \deg h_3, \quad \deg g = \deg h_2 + \deg h_4.$$

Hence

$$\deg f - \deg g = (\deg h_1 - \deg h_2) + (\deg h_3 - \deg h_4) = 2(\deg h_1 - \deg h_2)$$

is an even number, a contradiction. \square

Theorem 3.3. *For the 2-bridge knot $\mathfrak{b}(p, 3)$, $\Phi_w(x, z)$ is irreducible in $\mathbb{C}[x, z]$.*

We first show Theorem 3.3. By Proposition 3.1, $\Phi_w(x, z) = P(z) + x^2Q(z)R(z)$, where

$$\begin{aligned} P(z) &= S_d(z) - S_{d-1}(z), \\ Q(z) &= S_{\ell-1-\lfloor \frac{\ell}{2} \rfloor}(z) \left(S_{\lfloor \frac{\ell}{2} \rfloor}(z) - S_{\lfloor \frac{\ell}{2} \rfloor-1}(z) \right), \\ R(z) &= (2 - z)S_{d-\ell-1}(z). \end{aligned}$$

Since $\deg P - \deg QR = d - ((\ell - 1) + (d - \ell)) = 1$ is an odd number, by Lemma 3.2, $\Phi_w(x, z) \in \mathbb{C}[x, z]$ is irreducible if $\gcd(P(z), Q(z)R(z)) = 1$.

The following lemma is standard, see e.g. [15].

Lemma 3.4. *For $n \geq 1$, one has*

- (1) $S_n(z)$ is a monic polynomial of degree n whose n roots are exactly $2 \cos\left(\frac{j}{n+1}\pi\right)$, $1 \leq j \leq n$.
- (2) $S_n(z) - S_{n-1}(z)$ is a monic polynomial of degree n whose n roots are exactly $2 \cos\left(\frac{2j+1}{2n+1}\pi\right)$, $0 \leq j \leq n-1$.

Lemma 3.5. $\gcd\left(S_d(z) - S_{d-1}(z), S_{\lfloor \frac{\ell}{2} \rfloor}(z) - S_{\lfloor \frac{\ell}{2} \rfloor - 1}(z)\right) = 1$.

Proof. By Lemma 3.4 (2), it suffices to show that

$$(3.2) \quad \frac{2j+1}{2d+1} \neq \frac{2j'+1}{2\lfloor \frac{\ell}{2} \rfloor + 1}$$

where $0 \leq j \leq d-1$ and $0 \leq j' \leq \lfloor \frac{\ell}{2} \rfloor - 1$. It is easy to see that (3.2) holds true if $\gcd(2d+1, 2\lfloor \frac{\ell}{2} \rfloor + 1) = 1$. Recall that $d = \frac{p-1}{2}$ and $\ell = \lfloor \frac{p}{3} \rfloor$. Since $3(2\lfloor \frac{\ell}{2} \rfloor + 1) - (2d+1)$ is equal to either $3\ell - p$ or $3(\ell+1) - p$, and $3\ell - p = 3\lfloor \frac{p}{3} \rfloor - p$ is equal to either -1 or -2 (note that $\gcd(p, 3) = 1$), $3(2\lfloor \frac{\ell}{2} \rfloor + 1) - (2d+1)$ is equal to either ± 1 or ± 2 . It follows that $\gcd(2d+1, 2\lfloor \frac{\ell}{2} \rfloor + 1)$ is a divisor of 2. Since $2d+1$ is odd, we must have $\gcd(2d+1, 2\lfloor \frac{\ell}{2} \rfloor + 1) = 1$. \square

Lemma 3.6. $\gcd\left(S_d(z) - S_{d-1}(z), S_{\ell-1-\lfloor \frac{\ell}{2} \rfloor}(z)\right) = \gcd(S_d(z) - S_{d-1}(z), S_{d-\ell-1}(z)) = 1$.

Proof. The proof is similar to that of Lemma 3.5. \square

We now finish the proof of Theorem 3.3. From Lemmas 3.4, 3.5 and 3.6, we have $\gcd(P(z), Q(z)R(z)) = 1$. Hence Lemma 3.2 implies that $\Phi_w(x, z)$ is irreducible in $\mathbb{C}[x, z]$ for the two-bridge knot $\mathfrak{b}(p, 3)$ and this completes the proof of Theorem 3.3.

In the remainder of this section, we dedicate to the proof of Proposition 3.1. For $j = 1, \dots, d$, let

$$w_j = a^{\varepsilon_j} b^{\varepsilon_{j+1}} \dots a^{\varepsilon_{2d-j}} b^{\varepsilon_{2d+1-j}}.$$

Then $w_1 = w$ and $w_{j+1} = (w_j)' = w^{(j)}$. Let $u_j := w_{j+1}a^{\varepsilon_j}$ and $v_j := b^{\varepsilon_j}w_{j+1}$ for $j = 1, \dots, d$, where $w_{d+1} := 1$.

Lemma 3.7. (1) *If $\varepsilon_j = \varepsilon_{j+1}$, then*

$$\begin{aligned} \operatorname{tr} w_j &= z \operatorname{tr} w_{j+1} - \operatorname{tr} w_{j+2}, \\ x \operatorname{tr} u_j &= x^2 \operatorname{tr} w_{j+1} - x \operatorname{tr} u_{j+1}, \\ x \operatorname{tr} v_j &= x^2 \operatorname{tr} w_{j+1} - x \operatorname{tr} v_{j+1}. \end{aligned}$$

(2) *If $\varepsilon_j = -\varepsilon_{j+1}$, then*

$$\operatorname{tr} w_j = (z - x^2) \operatorname{tr} w_{j+1} - \operatorname{tr} w_{j+2} + x \operatorname{tr} u_{j+1} + x \operatorname{tr} v_{j+1}.$$

Proof. See [12, Proposition A.3]. \square

For the two-bridge knot $\mathfrak{b}(p, 3)$, we have $\varepsilon_j = 1$ if $j \leq \ell$ and $\varepsilon_j = -1$ if $\ell+1 \leq j \leq d$, where $\ell = \lfloor \frac{p}{3} \rfloor$.

Case 1: $\ell + 1 \leq j \leq d$. Since $\varepsilon_j = \varepsilon_{j+1}$, by Lemma 3.7,

$$\begin{aligned} \operatorname{tr} w_j &= z \operatorname{tr} w_{j+1} - \operatorname{tr} w_{j+2}, \\ x \operatorname{tr} u_j &= x^2 \operatorname{tr} w_{j+1} - x \operatorname{tr} u_{j+1}, \\ x \operatorname{tr} v_j &= x^2 \operatorname{tr} w_{j+1} - x \operatorname{tr} v_{j+1} \end{aligned}$$

Note that $\operatorname{tr} w_d = \operatorname{tr} a^{\varepsilon_d} b^{\varepsilon_{d+1}} = \operatorname{tr} a^{\varepsilon_d} b^{\varepsilon_d} = z$ and $\operatorname{tr} w_{d+1} = \operatorname{tr} 1 = 2$. Hence

$$\begin{aligned} \operatorname{tr} w_j &= T_{d+1-j}(z), \\ x \operatorname{tr} u_j &= x^2 (\operatorname{tr} w_{j+1} - \operatorname{tr} w_{j+2} + \cdots + (-1)^{d-1-j} \operatorname{tr} w_d) + (-1)^{d-j} x \operatorname{tr} u_d \\ &= x^2 (T_{d-j}(z) - T_{d-1-j}(z) + \cdots + (-1)^{d-1-j} T_1(z) + (-1)^{d-j}), \\ x \operatorname{tr} v_j &= x^2 (\operatorname{tr} w_{j+1} - \operatorname{tr} w_{j+2} + \cdots + (-1)^{d-1-j} \operatorname{tr} w_d) + (-1)^{d-j} x \operatorname{tr} v_d \\ &= x^2 (T_{d-j}(z) - T_{d-1-j}(z) + \cdots + (-1)^{d-1-j} T_1(z) + (-1)^{d-j}), \end{aligned}$$

where $\{T_n(z)\}_n$ are the Chebyshev polynomials defined by $T_0(z) = 2$, $T_1(z) = z$ and $T_{n+1}(z) = zT_n(z) - T_{n-1}(z)$. In particular,

$$\begin{aligned} \operatorname{tr} w_{\ell+1} &= T_{d-\ell}(z), \\ x \operatorname{tr} u_{\ell+1} &= x \operatorname{tr} v_{\ell+1} = x^2 (T_{d-1-\ell}(z) - T_{d-2-\ell}(z) + \cdots + (-1)^{d-\ell-2} T_1(z) + (-1)^{d-\ell-1}). \end{aligned}$$

Case 2: $1 \leq j \leq \ell - 1$. Since $\varepsilon_j = \varepsilon_{j+1}$, by Lemma 3.7,

$$\operatorname{tr} w_j = z \operatorname{tr} w_{j+1} - \operatorname{tr} w_{j+2}.$$

It follows that $\operatorname{tr} w_j = S_{\ell-j}(z) \operatorname{tr} w_{\ell} - S_{\ell-1-j}(z) \operatorname{tr} w_{\ell+1}$.

Case 3: $j = \ell$. Since $\varepsilon_{\ell} = -\varepsilon_{\ell+1}$, by Lemma 3.7,

$$\begin{aligned} \operatorname{tr} w_{\ell} &= (z - x^2) \operatorname{tr} w_{\ell+1} - \operatorname{tr} w_{\ell+2} + x \operatorname{tr} u_{\ell+1} + x \operatorname{tr} v_{\ell+1} \\ &= (z - x^2) T_{d-\ell}(z) - T_{d-\ell-1}(z) \\ &\quad + 2x^2 (T_{d-1-\ell}(z) - T_{d-2-\ell}(z) + \cdots + (-1)^{d-\ell-2} T_1(z) + (-1)^{d-\ell-1}). \end{aligned}$$

Hence $\Phi_w(x, z)$ is equal to

$$\begin{aligned} &\operatorname{tr} w_1 - \operatorname{tr} w_2 + \cdots + (-1)^{\ell-1} \operatorname{tr} w_{\ell} + (-1)^{\ell} \operatorname{tr} w_{\ell+1} + \cdots + (-1)^{d-1} \operatorname{tr} w_d + (-1)^d \\ &= (S_{\ell-1}(z) - S_{\ell-2}(z) + \cdots + (-1)^{\ell-2} S_1(z) + (-1)^{\ell-1} S_0(z)) \operatorname{tr} w_{\ell} \\ &\quad - (S_{\ell-2}(z) - S_{\ell-3}(z) + \cdots + (-1)^{\ell-2} S_0(z) + (-1)^{\ell-1} S_{-1}(z)) \operatorname{tr} w_{\ell+1} \\ &\quad + (-1)^{\ell} \operatorname{tr} w_{\ell+1} + \cdots + (-1)^{d-1} \operatorname{tr} w_d + (-1)^d \\ &= P(z) + x^2 Q(z) R(z), \end{aligned}$$

where

$$\begin{aligned} P(z) &= T_{d-\ell-2}(z) (S_{\ell-1}(z) - S_{\ell-2}(z) + \cdots + (-1)^{\ell-2} S_1(z) + (-1)^{\ell-1} S_0(z)) \\ &\quad - (S_{\ell-2}(z) - S_{\ell-3}(z) + \cdots + (-1)^{\ell-2} S_0(z) + (-1)^{\ell-1} S_{-1}(z)) T_{d-\ell}(z) \\ &\quad + (-1)^{\ell} T_{d-\ell}(z) + (-1)^{\ell+1} T_{d-\ell-1}(z) + \cdots + (-1)^{d-1} T_1(z) + (-1)^d, \\ Q(z) &= S_{\ell-1}(z) - S_{\ell-2}(z) + \cdots + (-1)^{\ell-2} S_1(z) + (-1)^{\ell-1} S_0(z), \\ R(z) &= -T_{d-\ell}(z) + 2 (T_{d-1-\ell}(z) - T_{d-2-\ell}(z) + \cdots + (-1)^{d-\ell-2} T_1(z) + (-1)^{d-\ell-1}). \end{aligned}$$

To proceed, we need the following lemma

Lemma 3.8. *The followings holds.*

$$(1) \ P(z) = S_d(z) - S_{d-1}(z),$$

$$(2) \quad Q(z) = S_{\ell-1-\lfloor \frac{\ell}{2} \rfloor}(z) \left(S_{\lfloor \frac{\ell}{2} \rfloor}(z) - S_{\lfloor \frac{\ell}{2} \rfloor-1}(z) \right),$$

$$(3) \quad R(z) = (2-z)S_{d-\ell-1}(z).$$

Proof. (1) is shown by [11] (see also [12, Proposition A.2]), $P(z) = \Phi_w(0, z) = S_d(z) - S_{d-1}(z)$.

To show (2), let

$$\alpha_n = S_n(z) - S_{n-1}(z) + \cdots + (-1)^{n-1}S_1(z) + (-1)^nS_0(z).$$

Then $Q(z) = \alpha_{\ell-1}$. If $n = 2k$ is even then

$$\begin{aligned} \alpha_n &= (S_{2k}(z) + \cdots + S_0(z)) - (S_{2k-1}(z) + \cdots + S_1(z)) \\ &= S_k^2(z) - S_k(z)S_{k-1}(z) = S_k(z)(S_k(z) - S_{k-1}(z)). \end{aligned}$$

If $n = 2k + 1$ is odd then

$$\begin{aligned} \alpha_n &= (S_{2k+1}(z) + \cdots + S_1(z)) - (S_{2k}(z) + \cdots + S_0(z)) \\ &= S_{k+1}(z)S_k(z) - S_k^2(z) = S_k(z)(S_{k+1}(z) - S_k(z)). \end{aligned}$$

In both case $\alpha_n = S_{n-\lfloor \frac{n+1}{2} \rfloor}(z) \left(S_{\lfloor \frac{n+1}{2} \rfloor}(z) - S_{\lfloor \frac{n+1}{2} \rfloor-1}(z) \right)$. Hence

$$Q(z) = \alpha_{\ell-1} = S_{\ell-1-\lfloor \frac{\ell}{2} \rfloor}(z) \left(S_{\lfloor \frac{\ell}{2} \rfloor}(z) - S_{\lfloor \frac{\ell}{2} \rfloor-1}(z) \right).$$

To show (3), let

$$\beta_n = -T_{n+1}(z) + 2(T_n(z) - T_{n-1}(z) + \cdots + (-1)^{n-1}T_1(z) + (-1)^n)$$

Then $R(z) = \beta_{d-\ell-1}$. Note that $T_j(z) = S_j(z) - S_{j-2}(z)$. If $n = 2k$ is even then

$$\begin{aligned} \beta_n &= -T_{2k+1}(z) + 2 + 2((T_{2k}(z) + \cdots + T_2(z)) - (T_{2k-1}(z) + \cdots + T_1(z))) \\ &= -(S_{2k+1}(z) - S_{2k-1}(z)) + 2 + 2((S_{2k}(z) - S_0(z)) - (S_{2k-1}(z) - S_{-1}(z))) \\ &= -(S_{2k+1}(z) + S_{2k-1}(z)) + 2S_{2k}(z) \\ &= (2-z)S_{2k}(z) = (2-z)S_n(z). \end{aligned}$$

If $n = 2k + 1$ is odd then

$$\begin{aligned} \beta_n &= -T_{2k+2}(z) - 2 + 2((T_{2k+1}(z) + \cdots + T_1(z)) - (T_{2k}(z) + \cdots + T_2(z))) \\ &= -(S_{2k+2}(z) - S_{2k}(z)) - 2 + 2((S_{2k+1}(z) - S_{-1}(z)) - (S_{2k}(z) - S_0(z))) \\ &= -(S_{2k+2}(z) + S_{2k}(z)) + 2S_{2k+1}(z) \\ &= (2-z)S_{2k+1}(z) = (2-z)S_n(z). \end{aligned}$$

In both cases $\beta_n = (2-z)S_n(z)$. Hence $R(z) = \beta_{d-\ell-1} = (2-z)S_{d-\ell-1}$. \square

From Lemma 3.8, we get

$$\begin{aligned} \Phi_w(x, z) &= P(z) + x^2Q(z)R(z) \\ &= S_d(z) - S_{d-1}(z) + x^2(2-z)S_{d-\ell-1}(z)S_{\ell-1-\lfloor \frac{\ell}{2} \rfloor}(z) \left(S_{\lfloor \frac{\ell}{2} \rfloor}(z) - S_{\lfloor \frac{\ell}{2} \rfloor-1}(z) \right). \end{aligned}$$

This completes the proof of Proposition 3.1.

4. AN APPLICATION TO A PARTIAL ORDERING OF PRIME KNOTS

Theorem 1.1 determines some minimal elements for a partial order on the set of prime knots in \mathbb{S}^3 defined as follows (refer to [10] for example). Let K and K' be prime knots in \mathbb{S}^3 which are *non-trivial*, i.e., they do not bound embedded disks in \mathbb{S}^3 . Then we write $K \geq K'$ if there exists an epimorphism (a surjective group homomorphism) from $G(K)$ onto $G(K')$. This defines a partial order on the set of prime knots. We can apply the following theorem to the partial order \geq .

Theorem 4.1 (Theorem 4.4 in [3], cf. Appendix in [15]). *Suppose $K \subset \mathbb{S}^3$ is a hyperbolic knot in \mathbb{S}^3 such that $X(K)$ of K has only one irreducible component that contains the characters of irreducible representations. Then $G(K)$ does not surject onto the knot group of any other non-trivial knot.*

We remark that Corollary 1.3 in [3] and Corollary 7.1 in [17] also give the following (compare to Theorem 4.1).

Theorem 4.2 (Corollary 7.1 in [17]). *Let K be a hyperbolic 2-bridge knot, K' a non-trivial knot. Suppose there exists an epimorphism $\phi : G(K) \rightarrow G(K')$. Then K' is a 2-bridge knot and furthermore $X(K)$ has more than one irreducible components containing the characters of irreducible representations.*

Since any twist knot is a 2-bridge knot, combining Theorems 1.1 and 4.1 (or Theorem 4.2), we obtain the following corollary.

Corollary 4.3. *For any integer $m \neq 0, 1$, at which the m -twist knot K_m is non-trivial, K_m is a minimal element for the partial order \geq .*

Note that Corollary 4.3 was shown by Kitano and Suzuki [10] in the case of K_1 , which is the trefoil knot (i.e., non-hyperbolic) and to which we cannot apply Theorems 4.1 and 4.2.

We can also apply Theorems 1.2 and 4.1 to get the following.

Corollary 4.4. *For any odd integer $p > 3$ satisfying $\gcd(p, 3) = 1$, where $\mathbf{b}(p, 3)$ is hyperbolic, $\mathbf{b}(p, 3)$ is a minimal element for the partial order \geq .*

Corollaries 4.3 and 4.4 also show the minimality of twist knots K_m ($m \geq 1$) and the 2-bridge knots $\mathbf{b}(p, 3)$ with respect to the partial order introduced by Silver and Whitten [22] (see also [8]).

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